

IRREDUCIBLY ODD GRAPHS

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ABSTRACT. An *irreducibly odd graph* is a graph such that each vertex has odd degree and for every pair of vertices, a third vertex in the graph is adjacent to exactly one of the pair. This family of graphs was introduced recently by Manturov [1] in relation to free knots. In this paper, we show that every graph is the induced subgraph of an irreducibly odd graph. Furthermore, we prove that irreducibly odd graphs must contain a particular minor called the *triskelion*.

1. INTRODUCTION

Manturov [1] defines a graph to be *odd* if every vertex $v \in G$ has odd degree and *irreducibly odd* if for any pair of vertices $u, v \in G$ there is a third vertex w adjacent to either u or v but not both.

Conversely, a graph is *reducible* if two vertices $u, v \in G$ have the same neighborhood (up to inclusion of one another). Such pairs of vertices are in a sense indistinguishable. For the duration of this paper, we consider only simple finite graphs.

We frequently refer to a degree 1 vertex as a *spike*, and call the graph below a *triskelion*. It is comprised of a bull graph with an additional spike at the “mouth.” This is the smallest non-empty irreducibly odd graph. In this paper, we will show that every irreducibly odd graph contains the triskelion as a minor, and that any graph can be augmented by spikes and bull graphs to produce an irreducibly odd graph.

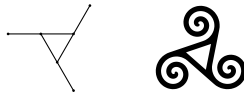


FIGURE 1. Two triskelions

The triskelion is the first in a family of graphs which we will call morningstar graphs, where the morningstar graph on $2n$ vertices is denoted M_n . It is easy to verify that each morningstar graph for $n \geq 3$ is irreducibly odd and has girth n . We prove the claim above by making a stronger observation: if a graph with *girth*, or minimal cycle length, k is irreducibly odd, then it contains M_k as a subgraph. This observation is proved in Section 3. We find that it is not only necessary to have a triskelion minor, but it is sufficient in a certain sense – bull graphs and spikes (heads and tails) may be attached to graphs to produce an irreducibly odd

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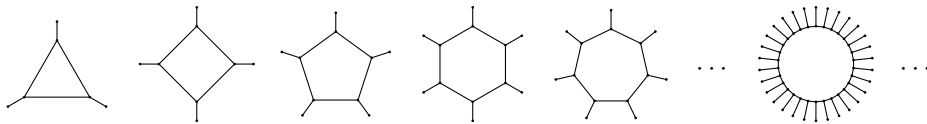


FIGURE 2. The family of morningstar graphs

graph. The details of this construction appear in Section 2. We note that every odd graph must have an even number of vertices, and in Section 4 we will show that the morningstar graphs are the smallest irreducibly odd graphs and other curios.

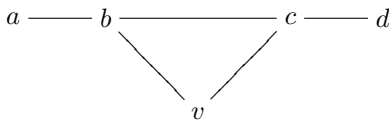
2. AUGMENTATION

Here we show that the family of irreducibly odd graphs is not minor-avoiding. The principal observation to make is that irreducible oddness can be considered to be a local property of graphs. That is, we say that a graph is irreducibly odd at a vertex v if that vertex has odd degree and for each $u \in G$ with $u \neq v$ there is a third vertex w with either $u \sim w$ or $v \sim w$.

Theorem 2.1. *If G is a graph with n vertices, then there is an irreducibly odd graph G' such that G is an induced subgraph of G' .*

Proof. Let G be a connected graph with at least two vertices and let some vertex $v \in G$ satisfy the property that there exists another vertex u such that any vertex adjacent to either u or v is adjacent to both. If v has even degree, then we can augment G by adding a vertex v' which is adjacent only to v , thereby creating a spike. Then, v' is adjacent to only v , so for any $w \in G$, there is a third vertex y with $w \sim y$ and $w \not\sim v'$ – the only obstruction to this would be another vertex only adjacent to v which cannot occur since any neighbor of v in the original graph must be adjacent to u . Hence, the augmented graph is irreducibly odd at v and v' .

If v has odd degree, we augment G by adding a copy of the bull graph



where a, b, c , and d are not already in G . As above, it is straightforward to verify that the augmented graph is irreducibly odd at a, b, c and d . Furthermore, v is adjacent to b and c and these are isolated from the rest of G so the augmented graph is irreducibly odd at v .

Note that the augmented graph is irreducibly odd at every vertex we have added, so this process need only be performed at most $\#V(G)$ times, adding at most four vertices at a time. We note that if G is not connected, we may introduce a single vertex adjacent to one vertex in each connected component before augmenting. \square

3. REQUIRED MINOR

Lemma 3.1. *No tree is irreducibly odd. That is, every irreducibly odd graph has a cycle.*

Proof. Note that the tree on two vertices does not have a third vertex, so is not irreducibly odd. Then, suppose that G is an odd tree with $n > 3$ vertices, and let

$$n_i = \# \{v \in G : \deg(v) = i\}$$

so G has n_1 leaves and max degree $m \leq n - 1$. Then, the number of edges in G is given by

$$n - 1 = \sum_{i=1}^m n_i - 1 = \frac{1}{2} \sum_{v \in G} \deg(v) = \frac{1}{2} \sum_{i=1}^m i n_i,$$

and interior vertices have degree at least 3, so

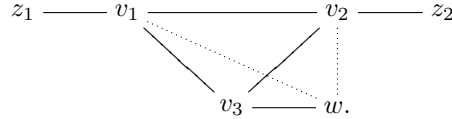
$$n_1 = 2 + \sum_{i=3}^m (i - 2)n_i \geq 2 + \sum_{i=3}^m n_i$$

and by the pigeonhole principle, some interior vertex v is adjacent to two leaves. Hence, each of these leaves is adjacent only to v and G is not irreducibly odd. \square

Theorem 3.2. *Every k -cycle in an irreducibly odd graph G with girth k belongs to a k -morningstar as a subgraph of G .*

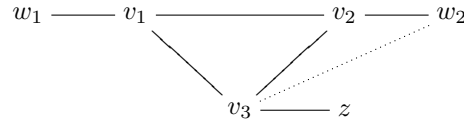
Proof. Let (v_1, v_2, \dots, v_k) be a cycle in an irreducibly odd graph G with girth $k \geq 3$. Since G is odd, $\deg v_i \geq 3$ so let $w_i \sim v_i$ with $w_i \notin \{v_{i-1}, v_{i+1}\}$ for each i . Since the cycle has minimal length, no $w_i \in \{v_1, \dots, v_k\}$. We denote the set $W = \{w_1, \dots, w_k\}$. If $\#W = k$, then there is a k -morningstar among $\{v_1, \dots, v_k, w_1, \dots, w_k\}$.

If $k = 3$ and $W = \{w\}$, then a clique is formed among $\{v_1, v_2, v_3, w\}$. By the irreducibility criterion, there exist vertices z_1 with $v_1 \sim z_1$ and $w \not\sim z_1$, and z_2 with $v_2 \sim z_2$ and $v_3 \not\sim z_2$ so we have the 3-morningstar



This picture is somewhat deceptive since we have not proven that $z_1 \neq z_2$. However, we note that each of v_1, v_2 is odd so has at least five neighbors which ensures freedom to choose different vertices z_1 and z_2 .

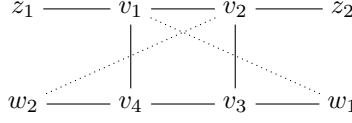
If instead we have $w_1 \neq w_2 = w_3$, so w_2 is adjacent to v_2 and v_3 , then by an irreducibility argument as above, we produce



where $z \not\sim v_2$ without loss of generality.

If $k = 4$, then $\#W \geq 2$ since a smaller set would imply the existence of triangles. Similarly, we note that adjacent vertices v_i, v_{i+1} cannot share a neighbor by the same reasoning. If we have $v_1, v_3 \sim w_1$ and $v_2, v_4 \sim w_2$ then as above, we produce vertices $z_1 \sim v_1$ and $z_1 \not\sim v_3$ and $z_2 \sim v_2$ and $z_2 \not\sim v_4$ by irreducibility to reveal the

4-morningstar



in G . As in the $k = 3, \#W = 1$ case, v_1 and v_2 have enough neighbors to ensure that $z_1 \neq z_2$ by the oddness condition. The $\#W = 3$ case can be examined similarly.

Finally, we note that if $k \geq 5$, no two vertices v_i, v_j can share a neighbor w . If $i - j \neq \pm 2 \pmod k$, then a shorter cycle is produced. Otherwise, the 4-cycle (v_i, v_{i+1}, v_j, w) is formed. Therefore, $\#W = k$ for all $k \geq 5$, producing the k -morningstar on $\{v_1, \dots, v_k, w_1, \dots, w_k\}$. \square

Corollary 3.3. *Every irreducibly odd graph contains the triskelion as a minor.*

Proof. It is easy to see that for each n , M_n is a minor of M_{n+1} . \square

4. MISCELLANEOUS PROPERTIES

Lemma 4.1. *Any irreducibly odd graph on $2k$ vertices has $2k \leq \#E(G) \leq \binom{2k}{2} - 2k + 1$, and these inequalities are sharp.*

Proof. The punchline of the proof of Lemma 3.1 was that no vertex may have two or more spikes. This indicates that at most half of the vertices in an irreducibly odd graph have degree 1, and that the rest have degree 3 or greater. That is, the smallest irreducibly odd graphs on $2k$ vertices have degree sequence $[3, \dots, 3, 1, \dots, 1]$. This degree sequence is realized by the k -morningstar though it is not uniquely the smallest.

To determine the largest irreducibly odd graphs, we introduce the notion of *irreducibly even* whose definition differs only by the parity of the vertex degrees – every vertex is even. It is easy to verify that if G is irreducibly odd, the complement \overline{G} is irreducibly even. Similarly, if G is irreducibly even and G has an even number of vertices, \overline{G} is irreducibly odd – but if G has an odd number of vertices, \overline{G} is also irreducibly even.

So, we note that an irreducibly even graph must have at most one isolated vertex (if there are two, the irreducibility criterion is not satisfied), and that every other vertex must have degree at least 2. Hence, the smallest irreducibly even graphs have degree sequence $[2, \dots, 2, 0]$. This is realized by the direct sum of a cycle of size $2k - 1$ and an isolated vertex, though not uniquely. The complement of such a graph has $\binom{2k}{2} - 2k + 1$ edges and, by construction, has the largest possible size for an irreducibly odd graph on $2k$ vertices. \square

We note that the proof of the girth $k \geq 5$ case of Theorem 3.2 does not require the irreducibility criterion. We find that we can weaken the irreducibility hypothesis significantly in this case.

Lemma 4.2. *If an odd graph G has girth $k \geq 5$ and no vertex has two spikes, G is irreducibly odd.*

Proof. Note that since G is odd, every vertex has a neighbor. Suppose that $u, v \in G$ and that $w \sim u$. In the case that $w \sim v$, then one of u or v has another neighbor y since w cannot have two spikes and (u, w, v) cannot be a cycle. Since we also have that (u, w, v, y) cannot be a cycle, either $u \not\sim y$ or $v \not\sim y$. Otherwise, $w \not\sim v$

straightaway, so every pair of vertices in G satisfies the irreducibility criterion hence G is irreducibly odd. \square

We see from Lemma 4.2 and Theorem 2.1 that any spike-free graph with girth $k \geq 5$ can be made irreducibly odd simply by adding spikes to even vertices. Additionally, it is easy to extend Theorem 2.1 to preserve the girth of the original graph (if it is not a forest), by adding copies of the k -morningstar less one spike instead of the bull graph.

Note that we have only considered finite graphs, but there do exist infinite irreducibly odd trees, for example, which obviously violate Lemma 3.1. In fact, any infinite odd tree with at most one leaf adjacent to any given vertex is irreducibly odd.

5. FREE KNOTS AND IRREDUCIBLY ODD GRAPHS

In 2009, Vassily Manturov introduced the concept of free knots [1], [2]. Free knots are equivalence classes of knot diagrams with two types of crossings: flat classical crossings and virtual crossings (denoted by an encircled flat crossing). Two diagrams of free knots are equivalent if they can be related by a sequence of flat classical and virtual Reidemeister moves as well as the flat virtualization move. See Figures 3 and 4.

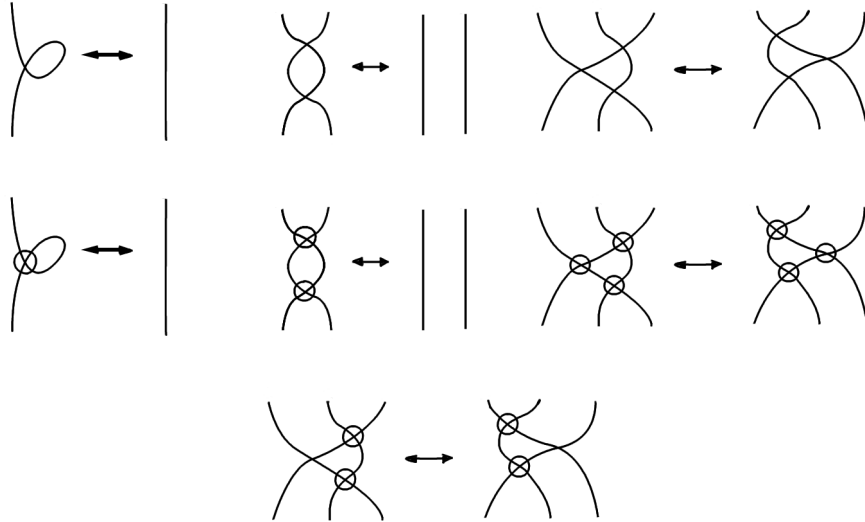


FIGURE 3. Flat classical and virtual Reidemeister moves

Note that the move pictured in Figure 5 is forbidden for free knots. We also note that all free knots with no virtual crossings are trivial.

It was shown in [1] that if a certain graph associated to a free knot is irreducibly odd, then the corresponding free knot diagram is minimal in the sense that it has the fewest number of classical crossings of all equivalent free knots. Thus, we can easily determine (classical) crossing numbers for such free knots. We recall in Figure 6 the simplest example of a free knot corresponding to an irreducibly odd

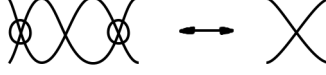


FIGURE 4. The flat virtualization move

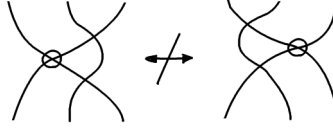


FIGURE 5. The forbidden move

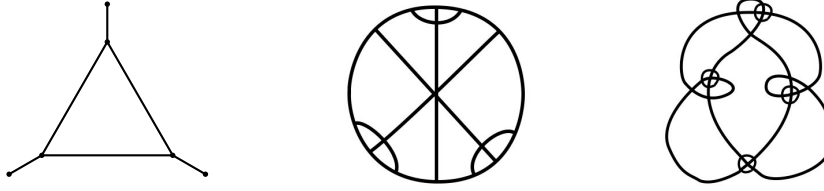


FIGURE 6. The triskelion, its chord diagram and its free knot diagram

graph. Furthermore, we note that this is the only example of a free knot with fewer than seven classical crossings that is associated to an irreducibly odd graph.

To find the graph associated to a free knot, it is useful to first draw the associated chord diagram. Given a free knot, we draw its chord diagram by first numbering the classical crossings in the diagram. We then, separately, draw a parametrizing (core) circle. We choose a base point on the knot and a corresponding base point on the circle. We then transverse the knot (in the direction of a chosen orientation) from the base point, keeping track along the circle of the order in which we encounter the crossings. Each crossing will appear twice along the circle, so we are able to form one chord in the circle corresponding to each crossing.

Once we have a chord diagram, we form our corresponding graph. Each chord in the chord diagram is associated with a vertex in our graph. There is an edge between two vertices if and only if the two corresponding chords in the chord diagram intersect.

In Figure 7, we provide another example of an irreducibly odd graph and its corresponding free knot diagram and chord diagram.

Clearly, every free knot has an associated graph. We note, however, that there are many graphs in general and irreducibly odd graphs in particular that do not have associated chord diagrams, and hence, cannot be associated to free knots. An

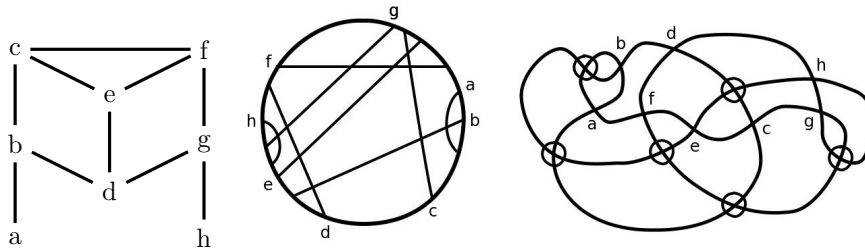


FIGURE 7. An example of a free knot with its corresponding chord diagram and graph

algorithm for determining whether a graph has an associated chord diagram (i.e. circle graph) can be found in [3].

6. FUTURE WORK

Our initial goal in studying irreducibly odd graphs was to begin the project of creating a free knot table. If every minimal free knot diagram corresponded to an irreducibly odd graph, this goal would be more tenable, since every free knot has a unique minimal diagram. There is another class of minimal free knot diagrams that is associated to a subclass of permutation graphs. We may study these objects to find a more complete picture of the theory of free knots. An introduction to permutation graphs and their relationship to free knots can be found in a paper by Boothby, Henrich, and Leaf, in preparation. Using these and other techniques, we aim to find a complete classification of small crossing free knots. However, it is unclear how to compute a complete table of minimal diagrams at this time.

The concept of reducibility should be investigated further. As we indicated in the introduction, two vertices having the same neighborhood are in a sense indistinguishable. This can be exploited, for example, in coloring a graph or computing its automorphism group.

APPENDIX A. IRREDUCIBLY ODD GRAPHS ON 6 AND 8 VERTICES

Table 1 contains all nonempty irreducibly odd graphs with 8 or fewer vertices. Chord diagrams are shown for circular graphs, i.e. those graphs that are associated to free knots.

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





















































			
			
			
			
			
			
			
			
			
			
			
			
			
			
			

TABLE 1. Irreducibly odd graphs on 6 and 8 vertices